Differential Geometry IV: General Relativity

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5.1 Let $\mathcal{M}$ be a differentiable manifold and ∇ a connection on $\mathcal{M}$.

(a) Show that there exists no $(1,2)$-type tensor field A on $\mathcal{M}$ with the property that, in any local coordinate system $(x^1,\dots,x^n)$ on $\mathcal{M}$

$$A^k_{ij} = \Gamma^k_{ij}.$$

Hint: Check how Γ^k_{ij} transforms under changes of coordinates.

(b) Show that the torsion $T : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M})$ of the connection ∇, which is defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

is a tensor field.

(c) Let $\bar{\nabla}$ be a (possibly) different connection on $\mathcal{M}$. Show that the difference $\nabla - \bar{\nabla} : \Gamma(\mathcal{M}) \times \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M})$ is also a tensor field. Deduce that, there exists a $(1,2)$-type tensor field A such that, in any given local coordinate system $(x^1,\dots,x^n)$,

$$A^k_{ij} = \Gamma^k_{ij} - \bar{\Gamma}^k_{ij}$$

where Γ^k_{ij} and $\bar{\Gamma}^k_{ij}$ are the Christoffel symbols of ∇ and $\bar{\nabla}$, respectively.

5.2 Let $\Psi : \mathcal{M}^n \to \mathbb{R}^{n+1}$ be an immersion such that $\Psi(\mathcal{M})$ is a *spacelike* hypersurface of $(\mathbb{R}^{n+1}, \eta)$ and let $\bar{g} = \Psi_* \eta$ be the induced metric. Let $(x^1,\dots,x^n)$ be a local coordinate chart on $\mathcal{M}$. Compute the Christoffel symbols Γ^k_{ij} of the Levi-Civita connection associated to $\bar{g}$ in the $(x^1,\dots,x^n)$ coordinates as functions of Ψ and its derivatives.

5.3 Let $\mathcal{M}$ be a smooth manifold equipped with a connection ∇. We can extend the connection ∇ to a map $\nabla : \Gamma(M) \times Ten^k_l(\mathcal{M}) \to Ten^k_l(\mathcal{M})$ by the requirements that

- ∇ satisfies the Leibniz rule with respect to tensor products, i.e. for all $X \in \Gamma(M)$

$$\nabla_X(f \otimes g) = \nabla_X f \otimes g + f \otimes \nabla_X g,$$

- ∇ commutes with contractions, i.e.

$$\nabla_X(\mathrm{tr}A) = \mathrm{tr}\big(\nabla_X A\big).$$

Show that, in any local coordinate chart $(x^1,\dots,x^n)$, if Γ^k_{ij} are the Christoffel symbols of ∇ then, for every 1-form ω:

$$(\nabla_{\frac{\partial}{\partial x^i}} \omega)_j = \partial_i \omega_j - \Gamma^k_{ij} \omega_k.$$

Moreover, for any (k,l)-tensor field T:

$$(\nabla_{\frac{\partial}{\partial x^a}} T)^{i_1\dots i_k}{}_{j_1\dots j_l} = \partial_a T^{i_1\dots i_k}{}_{j_1\dots j_l} + \Gamma^{i_1}_{ab} T^{b i_2\dots i_k}{}_{j_1\dots j_l} + \dots + \Gamma^{i_k}_{ab} T^{i_1\dots i_{k-1} b}{}_{j_1\dots j_l} - \Gamma^b_{a j_1} T^{i_1 i_2\dots i_k}{}_{b j_2\dots j_l} - \dots - \Gamma^b_{a j_l} T^{i_1\dots i_{k-1} i_k}{}_{j_1\dots j_{l-1} b}.$$

5.4 Let $(\overline{\mathcal{M}}, \bar{g})$ be a *Riemannian* manifold (i.e. $\bar{g}$ is positive definite) and let us define the Lorentzian manifold $(\mathcal{M}, g)$ so that $\mathcal{M} = \mathbb{R} \times \overline{\mathcal{M}}$ and g is the product metric $g = -(dt)^2 + \bar{g}$; this means that, for every local coordinate chart $(x^1, \ldots, x^n)$ on $\mathcal{U} \subset \overline{\mathcal{M}}$, if we extend it to a local coordinate chart $(t, x^1, \ldots, x^n)$ on $\mathbb{R} \times \mathcal{U} \subset \mathcal{M}$ so that t is simply the projection on the $\mathbb{R}$ factor, then

$$g = -dt^2 + \bar{g}_{ij} dx^i dx^j.$$

Show that a curve $\gamma : (0,1) \to (M, g)$ is a geodesic (for the Levi-Civita connection of g) if and only, in any local coordinate system $(t; x^1, \ldots, x^n)$ as above, if it can be written in the form

$$\gamma(s) = (t(s); \bar{\gamma}^i(s))$$

where $t(s) = \lambda_1 s + \lambda_0$ for some $\lambda_1, \lambda_0 \in \mathbb{R}$ and $\bar{\gamma} : (0,1) \to \overline{\mathcal{M}}$ is a geodesic of $(\overline{\mathcal{M}}, \bar{g})$.

5.5 In this exercise, we will prove that there exist compact Lorentzian manifolds which are **geodesically incomplete** (recall that, as a consequence of the Hopf–Rinow theorem in Riemannian geometry, every compact Riemannian manifold is geodesically complete). Consider the manifold $\mathcal{M} = \mathbb{R}^2 \setminus 0$ equipped with the metric

$$g = \frac{1}{u^2 + v^2} du dv.$$

(a) Verify that $(\mathcal{M}, g)$ is a smooth Lorentzian manifold and that the map $(u, v) \to (\lambda \cdot u, \lambda \cdot v)$ is an isometry for every $\lambda \neq 0$.

(b) Consider the group of isometries $\Gamma = \{(u, v) \to (2^k u, 2^k, v),\ k \in \mathbb{Z}\}$. Show that the quotient space $\mathcal{M}/\Gamma$ is a compact manifold. Show also that $\mathcal{M}/\Gamma$ inherits a natural metric $\tilde{g}$ from $(\mathcal{M}, g)$ so that the quotient map $(\mathcal{M}, g) \to (\mathcal{M}/\Gamma, \tilde{g})$ is a local isometry.

(c) Show that the map $(\mathcal{M}, g) \to (\mathcal{M}/\Gamma, \tilde{g})$ maps geodesics to geodesics. Compute the geodesic equation on $(\mathcal{M}, g)$ and deduce that $(\mathcal{M}/\Gamma, \tilde{g})$ contains a geodesic $\gamma : (a, b) \to \mathcal{M}/\Gamma$ with $b < +\infty$ which *cannot* be extended beyond $t = b$.